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Projective Unitary Positive-Energy Representations of Diff(S^1)

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Let $\mathscr{D}$ be the group of orientation-preserving diffeomorphisms of the circle S^1. Then $\mathscr{D}$ is Fréchet Lie group with Lie algebra $(\mathfrak{d}_\infty)_{\mathbb{R}}$ the smooth real vector fields on S^1. Let $\mathfrak{d}_{\mathbb{R}}$ be the subalgebra of real vector fields with finite Fourier series. It is proved that every infinitesimally unitary projective positive-energy representation of $\mathfrak{d}_{\mathbb{R}}$ integrates to a continuous projective unitary representation of $\mathscr{D}$. This result was conjectured by V. Kac. © 1985 Academic Press, Inc.

0. INTRODUCTION

Let $\mathscr{D}$ denote the group of orientation-preserving diffeomorphisms of the unit circle, S^1. Then $\mathscr{D}$ can be looked upon as a Frechet Lie group with Lie algebra $(\mathfrak{d}_\infty)_{\mathbb{R}}$, the smooth real vector fields on the circle. We denote by $\mathfrak{d}$ the Lie algebra of complex vector fields on the circle with finite Fourier series, and by $\mathfrak{d}_{\mathbb{R}}$ the real vector fiels in $\mathfrak{d}$. Let $\hat{\mathfrak{d}}$ be the central extension of $\mathfrak{d}$ defined by the cocycle ω (see Sect. 1 of this paper). We set

$$d_n = -ie^{in\theta}\frac{d}{d\theta}$$

and let κ be the central element for the extension. Then $\hat{\mathfrak{d}}$ has basis $\{d_n\}_{n\in\mathbb{Z}}$ and κ. The communication relations are

$$[d_n, d_m] = (m-n)\, d_{m+n} + \delta_{n,-m}\frac{n(n^2-1)}{12}\kappa \tag{0.1}$$

$$[d_n, \kappa] = 0. \tag{0.2}$$

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The Lie algebra $\hat{\mathfrak{d}}$ is also referred to as the *Virasoro* algebra in the physics literature.

Define the subalgebras

$$\mathfrak{h} = \mathbb{C}d_0 \oplus \mathbb{C}\kappa, \qquad \mathfrak{n} = \sum_{n>0} \mathbb{C}d_n, \qquad \mathfrak{n}^- = \sum_{n<0} \mathbb{C}d_n. \tag{0.3}$$

Then $\hat{\mathfrak{d}} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ is a triangular decomposition of $\hat{\mathfrak{d}}$ that one can use to form Verma (highest weight) modules for $\hat{\mathfrak{d}}$ and their irreducible quotients. The theory of these modules was initiated by Kac [Ka1], who introduced the analogue of the Shapavolov form and gave a marvelous formula for the determinant of this form.

Let us parametrize the irreducible highest weight modules for $\hat{\mathfrak{d}}$ by $L(h, c)$, where h is the highest weight of d_0 and c is the scalar by which κ acts. In [Ka2], Kac made the natural conjecture that if the contravariant (= Hermitian Shapavolov) form is positive definitive, then $L(h, c)$ "integrates" to a continuous projective representation of $\mathscr{D}$. In this paper we prove this conjecture. We note that the proof does not use the Kac determinant formula, nor does it rely on any classification of the values of (h, c) such that the contravariant form is positive definite.

The difficulty of proving the above results rests on the fact that the subgroup of $\mathscr{D}$ generated by the one-parameter subgroups is nowhere dense in $\mathscr{D}$. The standard methods of "integrating" a Lie algebra representation for finite-dimensional Lie algebras and groups do not apply in this situation. Furthermore, an appropriate $\mathscr{D}$-invariant domain of "C^∞ vectors" has to be constructed on which the actions of $\mathscr{D}$ and $\mathfrak{d}$ can be compared.

In this paper we overcome these difficulties by first extending the results of [G–W] concerning representations of loop algebras $\tilde{\mathfrak{g}}$ and their central extensions to the case $\mathfrak{g} = \mathbb{C}$ (the one-dimensional abelian Lie algebra). We then use the techniques of [G-W, Sect. 7] in this context to prove the integrability conjecture for $c = 1$ and $h \leqslant 0$. The argument for general (h, c) is quite delicate. The key step is the construction of a parametrized family of "Fock models" $(\Phi_{\lambda,\xi}, V)$, with V a fixed pre-Hilbert space and $\lambda, \xi \in \mathbb{C}$. These $\hat{\mathfrak{d}}$ modules have composition series by highest weight modules, and have the property that

$$\langle \Phi_{\lambda,\xi}(d)\, v, w \rangle = -\langle v, \Phi_{\bar{\lambda},\bar{\xi}}(d)\, w \rangle$$

for $d \in \mathfrak{d}_{\mathbb{R}}$ and $v, w \in V$. If $-h = \lambda^2/2 + \xi^2/2$ and $c = 1 + 12\xi^2$, then $L(h, c)$ is a canonical subquotient of $(\Phi_{\lambda,\xi}, V)$. Taking λ and ξ real, we thus find that the contravariant form on $L(h, c)$ is positive definite in the range $c \geqslant 1$ and $h \leqslant (1 - c)/24$. This continuous range of values of (h, c) with $L(h, c)$ realized as a pre-Hilbert subspace of the fixed space V is one of the critical ingredients in this paper. The other key ingredient is the *a priori* estimate for the action of $\mathfrak{d}$ in unitarizable modules which we obtain in Section 3. The rest of the arguments in the paper use tensor products and the techniques of [G-W].

We thank the referee for pointing out that the infinitesimal form of the Fock models appears in an article of Choros and Thorn [C-T] in connection with the quantization of string models, where the construction is attributed to Fairlie. These representations have also been realized by Neretin [Ner] via the Shale–Weyl representation (second-quantization) of a suitable symplectic action of $\mathscr{D}$.

We should point out that solutions of special cases of the integrability problem already appear in the literature. The case $c=1$ and $h=-m^2/4$, m an integer, was treated by Segal [S]. For $c \in \mathbb{Z}$, $c \geqslant 1$ and certain values of h, the problem was solved in our previous paper [G-W], in connection with unitary representations of the loop algebras $\tilde{\mathfrak{g}}$ and their central extensions $\hat{\mathfrak{g}}$, where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra. In [Ner], a solution is announced when the highest weight (h, c) lies in a "sawtooth" region which contains the set $\{h \leqslant -1/48,\ c \geqslant 1\}$; however, this region is not the full quadrant $\{h \leqslant 0,\ c \geqslant 1)$ in which $L(h, c)$ is known by [Ka2] to be unitarizable. By special constructions Neretin also obtains unitary representations for the isolated highest weights $(0, \frac{1}{2})$, and $(-\frac{1}{2}, \frac{1}{2})$. (Neretin's sign convention for the parameter h is the opposite of ours.)

1. Some Preliminaries

Let $\mathfrak{d}_\infty$ be the Lie algebra of all smooth vector fields on the circle. If $X \in \mathfrak{d}_\infty$, then

$$X = f(e^{i\theta}) \frac{d}{d\theta} \tag{1.1}$$

with f a smooth function on the circle. We denote by $\mathfrak{d}$ the subalgebra of all $X \in \mathfrak{d}_\infty$ such that f has a finite Fourier series. Let d_n be the vector field with $f(e^{i\theta}) = -ie^{in\theta}$. Then $\{d_n\}_{n \in \mathbb{Z}}$ is a basis for $\mathfrak{d}$, and the commutation relations in $\mathfrak{d}$ are

$$[d_n, d_m] = (m-n)\, d_{m+n}.$$

If X is given by (1.1) and $f(e^{i\theta}) = \sum a_n e^{in\theta}$, we set for $t \geqslant 0$

$$\|X\|_t = \sum_n (1+|n|)^t\, |a_n|. \tag{1.2}$$

We note that

$$\|[X, Y]\|_t \leqslant \|X\|_{t+1} \|Y\|_{t+1} \tag{1.3}$$

for all $t \geqslant 0$ and $X, Y \in \mathfrak{d}_\infty$. We also observe that the usual C^∞ topology on $\mathfrak{d}_\infty$ is given by the family of norms $\|\cdot\|_t$, $t \geqslant 0$.

We define on $\mathfrak{d}$ a canonical two-cocycle

$$\omega(d_n, d_m) = \delta_{n,-m} \frac{n(n^2-1)}{12}. \tag{1.4}$$

It is a simple, if tedious, exercise to show that ω does indeed define a cocycle on $\mathfrak{d}$. It can also be shown that up to a scalar multiple and coboundary, ω is the unique two-cocycle on $\mathfrak{d}$ such that ω is zero on $\mathbb{C}d_0 \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_{-1}$. We note that

$$|\omega(X, T)| \leqslant \|X\|_{3/2} \|Y\|_{3/2}. \tag{1.5}$$

Hence ω extends by continuity to a two-cocycle on $\mathfrak{d}_\infty$ which is explicitly given by the integral formula

$$\omega(X, Y) = \frac{i}{24\pi} \int_0^{2\pi} \left\{ \frac{d^2 f}{d\theta^2}(e^{i\theta}) + f(e^{i\theta}) \right\} \frac{dg}{d\theta}(e^{i\theta})\, d\theta. \tag{1.6}$$

when $X = f(e^{i\theta})(d/d\theta)$ and $Y = g(e^{i\theta})(d/d\theta)$.

As mentioned in the Introduction, it is customary to form the central extension $\hat{\mathfrak{d}}$ of $\mathfrak{d}$, in which the elements d_n have the communication relations (0.1), (0.2). We may also define the central extension $\hat{\mathfrak{d}}_\infty$ of $\mathfrak{d}_\infty$ using ω. However, in this paper we will for the most part use $\mathfrak{d}$ and $\mathfrak{d}_\infty$ and keep track of the cocycles.

On $\mathfrak{d}_\infty$ we define the anti-involution $X \to X^*$ by setting $d_n^* = d_{-n}$. That is,

$$\left(f(e^{i\theta}) \frac{d}{d\theta} \right)^* = -\overline{f(e^{i\theta})} \frac{d}{d\theta}. \tag{1.7}$$

Thus X is a real vector field if and only if $X^* = -X$.

Let (π, V) be a representation of $\hat{\mathfrak{d}}$ with $\pi(\kappa) = cI$, where $c \in \mathbb{C}$. Then we shall look upon (π, V) as a projective representation of $\mathfrak{d}$ with commutation relations

$$[\pi(X), \pi(Y)] = \pi([X, Y]) + c\omega(X, Y)\, I,$$

for $X, Y \in \mathfrak{d}$. We will call (π, V) a *highest weight* representation or a *positive-energy* representation if there is a complex number h such that

(i) the operator $\pi(d_0)$ diagonalizes on V with eigenvalues of the form $h-n$, $n \in \mathbb{N}$;

(ii) if $h-n$ is an eigenvalue, then the eigenspace V_{h-n} is finite dimensional;

(iii) $\dim(V_h)=1$.

We shall call the pair (h, c) the *highest weight* of the representation. A non-zero vector $v_0 \in V_h$ will be called a *highest weight vector*. Note that we do not assume that the highest weight vector is cyclic.

As is well known, for each $c \in \mathbb{C}$ and $h \in \mathbb{C}$ there exists an irreducible highest weight representation $(\pi_{h,c}, L(h, c))$, which is unique up to equivalence. This module is constructed as follows: Let $\mathfrak{h}$ and $\mathfrak{n}$ be as in (0.3) and define the subalgebra

$$\hat{\mathfrak{b}} = \mathfrak{h} \oplus \mathfrak{n}.$$

Let $\mathbb{C}_{h,c}$ be the $\hat{\mathfrak{b}}$-module $\mathbb{C}$ with action $\mathfrak{n} \cdot 1 = 0$, $\kappa \cdot 1 = c$, $d_0 \cdot 1 = h$. Set

$$M(h, c) = U(\hat{\mathfrak{d}}) \otimes_{U(\hat{\mathfrak{b}})} \mathbb{C}_{h,c}.$$

Here $U(\mathfrak{g})$ denotes the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Then it is easy to check that $M(h, c)$ is a highest weight representation with highest weight (h, c), and $M(h, c)$ has a unique non-zero irreducible quotient $L(h, c)$.

Let (π, V) be a projective $\mathfrak{d}$-module. Suppose $\langle \cdot, \cdot \rangle$ is a Hermitian form on V. Then this form is called *contravariant* if

$$\langle \pi(X) v, w \rangle = \langle v, \pi(X^*) w \rangle$$

for $X \in \mathfrak{d}$ and $v, w \in V$.

We note that if $h, c \in \mathbb{R}$, then $L(h, c)$ admits a unique (up to scalar multiple) nonzero contravariant form. This form is defined as follows: Let $\mathfrak{n}^-$ be defined by (0.3). Then $\hat{\mathfrak{d}} = \mathfrak{n}^- \oplus \hat{\mathfrak{b}}$. Thus by the Poincaré–Birkhoff–Witt theorem we have

$$U(\hat{\mathfrak{d}}) = U(\hat{\mathfrak{b}}) \oplus \mathfrak{n}^- \cdot U(\hat{\mathfrak{d}}).$$

Let p denote the projection of $U(\hat{\mathfrak{d}})$ onto $U(\hat{\mathfrak{b}})$ corresponding to this direct sum decomposition. We extend the anti-involution of $\mathfrak{d}$ to $\hat{\mathfrak{d}}$ by $\kappa^* = \kappa$, and then extend to $U(\hat{\mathfrak{d}})$ by $1^* = 1$ and $(xy)^* = y^* x^*$. Let

$$\xi_{h,c}\colon U(\hat{\mathfrak{b}}) \to \mathbb{C}$$

be the homomorphism corresponding to the action of $\hat{\mathfrak{b}}$ on $\mathbb{C}_{h,c}$. We can then define a Hermitian form on $U(\hat{\mathfrak{b}})$ by

$$(x, y)_{h,c} = \xi_{h,c}(p(y^*x)),$$

for $x, y \in U(\hat{\mathfrak{b}})$. One checks that if $h, c \in \mathbb{R}$, then

$$(x(z - \xi_{h,c}(z)), y)_{h,c} = (x, y(z - \xi_{h,c}(z))_{h,c} = 0$$

for $x, y \in U(\hat{\mathfrak{b}})$ and $z \in U(\hat{\mathfrak{b}})$. Thus $(\cdot, \cdot)_{h,c}$ pushes down to a contravariant form on $M(h, c)$. One then checks that the radical of this form is precisely the maximal proper submodule of $M(h, c)$. Thus one obtains a non-degenerate contravariant form on $L(h, c)$, denoted by $\langle \cdot, \cdot \rangle_{h,c}$.

In $\hat{\mathfrak{b}}$ there are many subalgebras isomorphic with $sl_2(\mathbb{C})$. We will be looking specifically at the subalgebras

$$\mathfrak{g}_n = \mathbb{C}d_n \oplus \mathbb{C}d_{-n} \oplus \mathbb{C}\left(d_0 - \frac{n^2-1}{24}\kappa\right),$$

for $n = 1, 2, \ldots$. We note that the real form

$$(\mathfrak{g}_n)_{\mathbb{R}} = \{X \in \mathfrak{g}_n \mid X^* = -X\}$$

is isomorphic with $sl_2(\mathbb{R})$ as a real Lie algebra.

On $sl_2(\mathbb{C})$ we define a conjugate-linear anti-involution by $X^* = -\bar{X}$, where $\bar{X}$ is the usual conjugation of matrices (no transpose). Thus $X^* = -X$ iff $X \in sl_2(\mathbb{R})$. We fix

$$H = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \qquad E = \tfrac{1}{2}\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}, \qquad F = \tfrac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}.$$

Then H, E, F have the standard commutation relations for the three-dimensional simple Lie algebra:

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

Furthermore, $H^* = H$ and $E^* = -F$.

If (π, V) is a highest weight module for $\hat{\mathfrak{b}}$ with highest weight (h, c) we can form for each $n = 1, 2, \ldots$, a representation (π_n, V) of $sl_2(\mathbb{C})$ by setting

$$\pi_n(E) = \frac{i}{n}\pi(d_n), \qquad \pi_n(F) = \frac{i}{n}\pi(d_{-n}) \tag{1.8}$$

and

$$\pi_n(H) = \frac{2}{n}\pi(d_0) - \frac{(n^2-1)}{12n}c. \tag{1.9}$$

If h and c are real and V carries a contravariant Hermitian form, then this form is also contravariant for π_n, relative to the anti-involution on $sl_2(\mathbb{C})$ defined above.

LEMMA 1.1 *Let $c, h \in \mathbb{R}$ and assume that $\langle \cdot, \cdot \rangle_{h,c}$ is positive definite on $L(h, c)$. Then either $h = c = 0$ and $L(0, 0)$ is the trivial module, or else $h \leqslant 0$ and $c > 0$.*

Proof. Let $v_0 = 1 \otimes 1 \in L(h, c)$ be the highest weight vector. Denote the action of $x \in U(\mathfrak{d})$ on $v \in L(h, c)$ by $x \cdot v$, and write $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_{h,c}$. Then from the commutation relations (0.1) we have

$$\begin{aligned}\langle d_{-n} \cdot v_0, d_{-n} \cdot v_0 \rangle &= \langle d_n d_{-n} \cdot v_0, v_0 \rangle \\ &= -2nh + \frac{n(n^2-1)}{12} c,\end{aligned}$$

for $n = 1, 2, \ldots$. Since the left side of this equation is nonnegative, we see that $h \leqslant 0$ by taking $n = 1$, and we see that $c \geqslant 0$ by letting $n \to \infty$.

Suppose $c = 0$. Set $x_n = (d_{-n})^2 \cdot v_0$ and $y_n = d_{-2n} \cdot v_0$, for $n = 1, 2, \ldots$. Then from the calculation above one has $\langle y_n, y_n \rangle = -4nh$. One calculates that

$$\begin{aligned}\langle x_n, y_n \rangle &= \langle d_{-n} \cdot v_0, d_n d_{-2n} \cdot v_0 \rangle \\ &= -3n \langle d_{-n} \cdot v_0, d_{-n} \cdot v_0 \rangle = 6n^2 h\end{aligned}$$

and

$$\begin{aligned}\langle x_n, x_n \rangle &= \langle d_{-n} \cdot v_0, d_{-n} d_n d_{-n} \cdot v_0 \rangle \\ &\quad - 2n(h-n) \langle d_{-n} \cdot v_0, d_{-n} \cdot v_0 \rangle \\ &= -2n(2h-n) \langle d_{-n} \cdot v_0, d_{-n} \cdot v_0 \rangle \\ &= -4n^2 h(n-2h).\end{aligned}$$

From these formulas one finds that the Gram determinant

$$\det \begin{bmatrix} \langle x_n, x_n \rangle & \langle x_n, y_n \rangle \\ \langle y_n, x_n \rangle & \langle y_n, y_n \rangle \end{bmatrix} = -16n^3 h^2 (5n + 8h).$$

Since the form $\langle \cdot, \cdot \rangle$ is assumed to be positive-definite, this determinant is nonnegative for all positive integers n. By letting $n \to \infty$, we see that h must be zero. ∎

2. A Priori Estimates in the Category $\mathscr{U}$

We retain the notation of the previous section. We shall denote by $\mathscr{U}$ the category of all *unitarizable* highest weight $\hat{\mathfrak{d}}$ modules (π, V). That is, V satisfies conditions Section 1 (i), (ii), and (iii), and has a positive-definite contravariant form $\langle \cdot, \cdot \rangle$. If $W \subset V$ is a submodule, then so is $W^{\perp}$ (orthogonal complement relative to $\langle \cdot, \cdot \rangle$), and both submodules are the direct sum of finite-dimensional d_0-eigenspaces. From this the following properties of the category $\mathscr{U}$ are easily verified:

(a) If $(\pi, V) \in \mathscr{U}$ has highest weight (h, c) and highest weight vector v_0, then the cyclic submodule $U(\hat{\mathfrak{d}}) \cdot v_0$ is irreducible and isomorphic to $L(h, c)$. The contravariant form on V restricts to a positive multiple of the contravariant form on $L(h, c)$

(b) If $(\pi, V) \in \mathscr{U}$ has highest weight (h, c), and p is a positive integer, then the p-fold tensor product $(\pi^{\otimes p}, V^{\otimes p}) \in \mathscr{U}$ and has highest weight (ph, pc).

(c) If $(\pi, V) \in \mathscr{U}$ and (π_n, V) is the representation of $sl_2(\mathbb{C})$ defined by (1.8) and (1.9), then V is the orthogonal direct sum of irreducible modules for $sl_2(\mathbb{C})$ (which are highest weight modules relative to the canonical basis $\{E, F, H\}$).

We note from (a) and Lemma 1.1 that the highest weights of modules in $\mathscr{U}$ satisfy $h \leqslant 0$ and $c \geqslant 0$.

For the rest of this section, we fix a module $(\pi, V) \in \mathscr{U}$, and denote by H_0 the Hilbert space completion of V relative to $\langle \cdot, \cdot \rangle$. Let A denote the operator on H_0 that is the closure of $I - \pi(d_0)$ on V. Since $d_0^* = d_0$, we see by the contravariance of $\langle \cdot, \cdot \rangle$ that A is a positive self-adjoint operator with discrete spectrum contained in the set $\{1 - h + n \mid n \in \mathbb{N}\}$, and the eigenspaces of A are finite-dimensional.

Let H_t for $t \in \mathbb{R}$ denote the Hilbert space completion of V relative to the inner product

$$\langle v, w \rangle_t = \langle A^t v, A^t w \rangle.$$

The form $\langle \cdot, \cdot \rangle$ then extends coninuously to a sesquilinear pairing of H_t and H_{-t}. We set

$$\|v\|_t = \|A^t v\| = \{\langle v, v \rangle_t\}^{1/2}.$$

for $v \in H_t$. Let

$$H_{\infty}(\pi) = \bigcap_{t \geqslant 0} H_t \qquad \text{and} \qquad H_{-\infty}(\pi) = \bigcup_{t \leqslant 0} H_t.$$

Give $H_\infty(\pi)$ the usual Fréchet topology from the norms $\|\cdot\|_t$, and give $H_{-\infty}(\pi)$ the inductive limit topology. The form $\langle\cdot,\cdot\rangle$ then extends by continuity to a nonsingular pairing between these spaces.

We now come to the main result of this section.

PROPOSITION 2.1. *If $X \in \mathfrak{d}$ and $v \in V$, then for all $t \in \mathbb{R}$,*

$$\|\pi(X)\, v\|_t \leqslant 2^{1/2}\|X\|_{|t|}\|v\|_{t+1} + M\|X\|_{|t|+1}\|v\|_{t+1/2} + M\|X\|_{|t|+3/2}\|v\|_t, \tag{2.1}$$

where $M = (c/12)^{1/2}$.

Proof. We shall use a modification of the technique in [G-W, Sect. 3]. Let $n \in \mathbb{Z}$, $n \geqslant 1$, and take the representation π_n of $\mathfrak{g} = sl_2(\mathbb{C})$ on V as in Section 1. By (c) and the subadditivity of estimate (2.1) in v and X, it will suffice to prove (2.1) when $X = d_n$ or d_{-n} and v satisfies the following:

(α) There exists a subspace $W \subset V$ containing v such that $\pi_n|_W$ is an irreducible highest weight representation of $\mathfrak{g}$, and there exists a real number $a \geqslant 1$ such that $Av = av$.

Let $\{E, F, H\}$ be the basis for $\mathfrak{g}$ as in Section 1, and let μ be the highest weight of $\pi_n(H)$ on W. Then

$$\pi_n(H)\, v = (\mu - 2m)\, v \tag{2.2}$$

for some $m \in \mathbb{N}$. Thus the Casimir operator

$$\Omega = \tfrac{1}{2} H^2 + EF + FE = \tfrac{1}{2} H^2 + H + 2FE \tag{2.3}$$

acts on W by the scalar $\frac{1}{2}\mu^2 + \mu$. Combining (2.2) and (2.3) we have

$$-\pi_n(F)\, \pi_n(E)\, v = m(m - \mu - 1)\, v.$$

Thus from the contravariance of $\langle\cdot,\cdot\rangle$, we obtain

$$\|\pi_n(E)\, v\|^2 = \langle \pi_n(F)\, \pi_n(E)\, v, v\rangle = m(m - \mu - 1)\,\|v\|^2.$$

In terms of the action of $\mathfrak{d}$, this gives the identity

$$\|\pi(d_n)\, v\|^2 = nm(nm - n\mu - n)\,\|v\|^2. \tag{2.4}$$

To obtain an estimate in terms of n and a from (2.4), we observe from (1.9) that

$$n(\mu - 2m) = 2(1 - a) - \frac{n^2 - 1}{12}\, c. \tag{2.5}$$

Also by Section 1 (i), there exists $k \in \mathbb{N}$ such that

$$n\mu = 2(h-k) - \frac{n^2-1}{12} c. \tag{2.6}$$

Subtracting (2.6) from (2.5) gives

$$nm = h - k + a - 1.$$

Since $h \leqslant 0$ and $nm \geqslant 0$, we see that

$$nm \leqslant a-1 \qquad \text{and} \qquad k-h \leqslant a-1.$$

Adding (2.5) and (2.6) and using this last inequality gives

$$\begin{aligned} nm - n\mu &= k - h + a - 1 + \frac{n^2-1}{12} c. \\ &\leqslant 2a + \frac{n^2-1}{12} c. \end{aligned}$$

Thus from (2.4) we obtain the estimate

$$\|\pi(d_n)\, v\|^2 \leqslant \left\{ 2a^2 + \frac{n^2-1}{12} ca \right\} \|v\|^2 \tag{2.7}$$

for all $n \geqslant 0$. Observe next that

$$\begin{aligned} \|\pi(d_{-n})\, v\|^2 &= \langle \pi(d_n)\, \pi(d_{-n})\, v, v \rangle \\ &= \frac{n^2-1}{12} cn + \langle \pi(d_{-n})\, \pi(d_n)\, v, v \rangle \\ &= \frac{n^2-1}{12} cn + \|\pi(d_n)\, v\|^2. \end{aligned}$$

Together with (2.7), this gives the basic estimate

$$\|\pi(d_n)\, v\| \leqslant \{ 2^{1/2} a + M(1+|n|)\, a^{1/2} + M(1+|n|)^{3/2} \} \|v\| \tag{2.8}$$

for all $n \in \mathbb{Z}$ and all $v \in V$ with $Av = av$, where $M = (c/12)^{1/2}$.

To pass from (2.8) to an estimate in the norm $\|\cdot\|_t$, we observe that if $n > a-1$, then $\pi(d_n)\, v = 0$. If $n \leqslant a-1$ and $t \in \mathbb{R}$, then $A^t \pi(d_n)\, v = (a-n)^t \pi(d_n)\, v$. Hence from (2.8) we have

$$\|A^t \pi(d_n)\, v\| \leqslant (a-n)^t \{ 2^{1/2} a + M(1+|n|)\, a^{1/2} + M(1-|n|)^{3/2} \} \|v\| \tag{2.9}$$

when $n \leqslant a-1$. The left side of (2.9) is zero when $n > a-1$.

We now prove (2.1) for $X=d_n$ and v as in (α) above by cases, depending on the signs of n and t. We may assume that $n \leqslant a-1$.

Case (a). $t \geqslant 0$ and $n \geqslant 0$. Then $(a-n)^t \leqslant a^t$. Since $\|v\|_t = a^t\|v\|$ and $\|d_n\|_t = (1+|n|)^{|t|}$, we immediately obtain (2.1) from (2.9).

Case (b). $t<0$ and $n \geqslant 0$. Then $(a-n)(1+n) \geqslant a$, so

$$(a-n)^t \leqslant (1+n)^{-t} a^t.$$

Using this in (2.9) yields (2.1).

Case (c). $t \geqslant 0$ and $n<0$. Since $a \geqslant 1$, we may use the inequality

$$(a-n)^t \leqslant a^t(1+|n|)^t$$

in (2.9) to obtain (2.1).

Case (d). $t<0$ and $n<0$. Since $a-n \geqslant$ a, we have

$$(a-n)^t \leqslant a^t.$$

Using this in (2.9) gives (2.1) and completes the proof of the theorem. ∎

COROLLARY 2.2. *(π, V) extends to continuous projective representations of $\mathfrak{d}_\infty$ on $H_\infty(\pi)$ and $H_{-\infty}(\pi)$, such that*

$$\langle \pi(X)\, v, w \rangle = \langle v, \pi(X^*)\, w \rangle$$

for $v \in H_\infty(\pi)$ and $w \in H_{-\infty}(\pi)$. If $X \in \mathfrak{d}_\infty$ and $X^*=X$, then $\pi(X)$, as an unbounded operator on H_0, is essentially self-adjoint on $H_\infty(\pi)$.

Proof. The extension of π by continuity to $\mathfrak{d}_\infty$ is immediate from (2.1). Furthermore, we have

$$\pi(X)\colon H_t \to H_{t-1} \tag{2.10}$$

continuously, for all $t \in \mathbb{R}$ and $X \in \mathfrak{d}_\infty$. Suppose $X^*=X$. Then $\pi(X)$ is a symmetric operator on $H_\infty(\pi)$. Since $[A, \pi(X)] = \pi(Y) + \lambda I$, where $Y = [X, d_0] \in \mathfrak{d}_\infty$ and $\lambda = c\omega(X, d_0)$, we see from (2.10) that the hypotheses of Nelson's commutator theorem [Ne] (cf. [R-S; Theorem X.36]) are satisfied. This theorem implies the essential self-adjointness of $\pi(X)$. ∎

Remark. Let X be a real smooth vector field on S^1. From the corollary we see that there is a unitary representation of the one-parameter subgroup $\exp(tX)$ of $\mathscr{D}$ on H_0 whose infinitesimal generator is the closure of $\pi(X)$. We shall denote this one-parameter unitary group as $e^{it\pi(X)}$.

As remarked in the Introduction, the property of integrability along one-parameter subgroups of $\mathscr{D}$ does not lead directly to a representation of $\mathscr{D}$. As one step in our construction of such a representation, we use the *a priori* estimates to obtain the following perturbation result for the energy operator $\pi(d_0)$:

THEOREM 2.3 *Let $\Omega \subset \mathbb{R}^2$ be a compact set. Then there is a neighborhood $\mathscr{M}$ of d_0 in $\mathfrak{d}_\infty$, depending only on Ω, such that if $X \in \mathscr{M}$, $X = X^*$, and $(\pi, V) \in \mathscr{U}$ has highest weight $(h, c) \in \Omega$, then the following properties hold*:

(i) *If $w \in H_{-\infty}(\pi)$ and $\pi(X)\, w \in H_t$, then $w \in H_{t+1}$.*

(ii) *The operator $D(X) = \pi(X)|_{H_1}$ is self-adjoint as an unbounded operator on H_0. The resolvent of $D(X)$ maps H_t onto H_{t+1} for all $t \in \mathbb{R}$ and is a compact operator on H_0. In particular,*

$$\bigcap_{n \geq 0} \operatorname{Dom}(D(X)^n) = H_\infty(\pi)$$

and is independent of X.

(iii) *The operator $D(X)$ is bounded above. The highest eigenvalue $\mu_0(X)$ has multiplicity one and depends continuously on X. Furthermore, $|\mu_0(X) - h| < \frac{1}{2}$ and all other eigenvalues of $D(X)$ are less than $h - \frac{1}{2}$.*

Proof. This result is proved by exactly the same argument as that of [G-W; Theorem 3.5] using Proposition 2.1 in place of [G-W; Lemma 3.3]. To obtain estimates uniformly over Ω, note that in estimate (2.1) the top order term involving $\|v\|_{t+1}$ is independent of the parameter c, and the constant M in the lower order terms is a continuous function of $c \geq 0$. ∎

3. SOME FOCK MODELS

Consider the Lie algebra $\hat{\mathfrak{g}}$ with basis z and $u(n)$, $n \in \mathbb{Z}$, and commutation relations

$$[u(n), u(m)] = n\delta_{n,-m} z, \qquad [z, u(n)] = 0,$$

for $m, n \in \mathbb{Z}$. Then in the notation of [G-W; Sect. 1.1], $\hat{\mathfrak{g}} = \hat{\mathbb{C}}$, where $\mathbb{C}$ is the abelian one-dimensional Lie algebra and $u = 1$. Note that $u(0)$ is central in $\hat{\mathfrak{g}}$. We set $1^* = 1$ on $\mathbb{C}$ and extend this to an anti-involution on $\hat{\mathfrak{g}}$ by setting $z^* = z$ and $u(n)^* = u(-n)$.

Given $\lambda \in \mathbb{C}$, we define the analogue of the standard module for $\hat{\mathfrak{g}}$ as follows: Let $V = \mathbb{C}[x_1, x_2, \ldots]$ be the polynomial algebra in an infinite number of variables. Set $\Phi_\lambda(u(0)) = \lambda I$, $\Phi_\lambda(z) = I$, and for $n > 0$,

$$\Phi_\lambda(u(n)) = n\frac{\partial}{\partial x_n}, \qquad \Phi_\lambda(u(-n)) = \text{multiplication by } x_n,$$

as operators on V. Then it is easy to check that (Φ_λ, V) is a module for $\hat{\mathfrak{g}}$, and that if $v \in V$ then $\Phi_\lambda(u(n))\, v = 0$ for n sufficiently large.

We put the inner product on V given on monomials by

$$\langle x^I, x^J \rangle = \delta_{I,J} \prod_{k=1}^{\infty} i_k! \prod_{k=1}^{\infty} k^{i_k}.$$

Here we use the multi-index notation

$$x^I = x^{i_1} x^{i_2} \cdots,$$

where $I = (i_1, i_2, \ldots)$ with $i_k = 0$ for k sufficiently large. (By convention, $0! = 1$, so the infinite products appearing above have all but a finite number of factors equal to 1.) Then one has

$$\langle \Phi_\lambda(x)\, v, w \rangle = \langle v, \Phi_{\bar\lambda}(x^*)\, w \rangle$$

for $x \in \hat{\mathfrak{g}}$ and $v, w \in V$. Of course, the operators $\Phi_\lambda(u(n))$ are independent of λ for $n \neq 0$.

We put on $\hat{\mathfrak{g}}$ the norms

$$\|x\|_t = |\alpha| + \sum_{n \in \mathbb{Z}} (1 + |n|)^t |a_n|,$$

where $t > 0$ and $x = \alpha z + \sum a_n u(n)$. We use $(\hat{\mathfrak{g}})_t$ to denote the completion of $\hat{\mathfrak{g}}$ relative to $\|\cdot\|_t$. We set

$$\hat{\mathfrak{g}}_\infty = \bigcap_{t \geqslant 0} (\hat{\mathfrak{g}})_t.$$

It is easily checked that if $t \geqslant \frac{1}{2}$ then

$$\|[X, Y]\|_t \leqslant \|X\|_t \|Y\|_t \tag{3.1}$$

(see [G-W; Lemma 3.1]), so that $(\hat{\mathfrak{g}})_t$ is a Banach Lie algebra.

We define an action of $\mathfrak{d}$ on $\hat{\mathfrak{g}}$ by $d_m \cdot u(n) = nu(u+m)$, for $n, m \in \mathbb{Z}$, and $d_m \cdot z = 0$ Then we have

$$\|d \cdot x\|_t \leqslant \|d\|_t \|x\|_{t+1} \tag{3.2}$$

for $d \in \mathfrak{d}$, $x \in \hat{\mathfrak{g}}$, and $t \geqslant 0$ (cf. [G-W; Lemma 3.1]). Using this action, we form the semi-direct product Lie algebra $\mathfrak{d} \ltimes \hat{\mathfrak{g}}$. From (3.2) we see that we can also form the semi-direct product $\mathfrak{d}_\infty \ltimes (\hat{\mathfrak{g}})_\infty$ and it is a Fréchet Lie algebra.

We now extend (Φ_λ, V) to a projective representation of $\mathfrak{d} \ltimes \hat{\mathfrak{g}}$ using formulas which first occured in the quantization of "string models" (cf. [C-T] and the references cited there). If $m \in \mathbb{Z}$ we set

$$T_\lambda(2m+1) = \sum_{k=0}^{\infty} \Phi_\lambda(u(m-k))\, \Phi_\lambda(u(m+k+1)),$$

$$T_\lambda(2m) = \tfrac{1}{2}\, \Phi_\lambda(u(m))^2 + \sum_{k=1}^{\infty} \Phi_\lambda(u(m-k))\, \Phi_\lambda(u(m+k)).$$

Since $\Phi_\lambda(u(n))\, v = 0$ for $v \in V$ and n large these formulas make sense on V. Arguing as in [G-W; Sect. 2] one verifies that

$$[\Phi_\lambda(u(p)),\ T_\lambda(q)] = pu(p+q), \tag{3.3}$$

$$T_\lambda(0) \cdot 1 = \lambda^2/2, \tag{3.4}$$

$$[T_\lambda(p),\ T_\lambda(q)] = (p-q)\, T_\lambda(p+q) + \delta_{p,-q} \frac{p(p^2-1)}{12}. \tag{3.5}$$

We set $\Phi_\lambda(d_p) = -T_\lambda(p)$ and extend Φ_λ by linearity to $\mathfrak{d}$. Then (3.5) implies that Φ_λ is a projective representation of $\mathfrak{d}$.

We note that

$$\Phi_\lambda(d_0) = -\frac{1}{2}\lambda^2 - \sum_{n=1}^{\infty} n x_n \frac{\partial}{\partial x_n}. \tag{3.6}$$

Thus the eigenvalues of $\Phi_\lambda(d_0)$ are $-\lambda^2/2 - n$, with $n \in \mathbb{N}$, and have finite multiplicity. The eigenspace for $-\lambda^2/2$ is spanned by 1. Hence Φ_λ is a highest weight representation of $\hat{\mathfrak{d}}$ with highest weight $(-\lambda^2/2, 1)$. Furthermore,

$$\langle \Phi_\lambda(d)\, v, w \rangle = \langle v, \Phi_{\bar{\lambda}}(d^*)\, w \rangle \tag{3.7}$$

for all $d \in \hat{\mathfrak{d}}$. In particular, when $\lambda \in \mathbb{R}$, then (Φ_λ, V) is in the category $\mathscr{U}$. From Section 2(a) and 2(b), this construction thus gives the following result:

Proposition 3.1. *If $h \leqslant 0$ and $n = 1, 2, \ldots$, then the contravariant form on $L(h, n)$ is positive-definite.*

We shall need *a priori* estimates for the action of $\mathfrak{d} \ltimes \hat{\mathfrak{g}}$ on V with control over the parameter λ. For this, we define a scale of Hilbert spaces using the fixed operator

$$A = I + \sum_{n=1}^{\infty} n x_n \frac{\partial}{\partial x_n}.$$

Note from (3.6) that $A = I - \Phi_\lambda(d_0) - \lambda^2/2$. Let H_0 denote the Hilbert space completion of V relative to $\langle \cdot, \cdot \rangle$. Just as in Section 2, the closure of A defines an unbounded positive self-adjoint operator on H_0, and we define the scale $(H_t, \|\cdot\|_t)$ of Hilbert spaces relative to A:

$$\langle v, w \rangle_t = \langle A^t v, A^t w \rangle.$$

Up to equivalent norms, these are the same spaces associated with the representation (Φ_λ, V) in Section 2 when $\lambda \in \mathbb{R}$.

LEMMA 3.2. *Let $\lambda \in \mathbb{C}$ and $t \in \mathbb{R}$. Then the following estimates hold:*

(i) *If $x \in \hat{\mathfrak{g}}$ and $v \in V$, then*

$$\|\Phi_\lambda(x)\, v\|_t \leqslant \|x\|_{|t|+1/2} \|v\|_{t+1/2} + |\lambda|\, \|x\|_{|t|} \|v\|_t.$$

(ii) *If $d \in \mathfrak{d}$ and $v \in V$ then*

$$\|\Phi_\lambda(d)\, v\|_t \leqslant 2^{1/2} \|d\|_{|t|} \|v\|_{t+1} + M \|d\|_{|t|+1} \|v\|_{t+1/2} + M \|d\|_{|t|+3/2} \|v\|_t + |\lambda|\, \|d\|_{|t|+1/2} \|v\|_{t+1/2} + |\lambda|^2 \|d\|_{|t|} \|v\|_t,$$

where $M = (1/12)^{1/2}$.

Proof. (i) We note that if $n > 0$ then

$$\Phi_\lambda(u(n))\, x^I = n i_n x^{I - \delta_n},$$

where δ_n is the infinite multi-index $(\delta_{1,n}, \delta_{2,n}, \ldots)$. Thus

$$\|\Phi_\lambda(u(n))\, x^I\| = n^{1/2} (i_n)^{1/2} \|x^I\|.$$

Also $\Phi_\lambda(u(-n))\, x^I = x_n\, x^I$, so

$$\|\Phi_\lambda(u(-n))\, x^I\| = n^{1/2} (1 + i_n)^{1/2} \|x^I\|.$$

Now $Ax^I = ax^I$, with $a = 1 + \sum n i_n$. For any $I \neq 0$, we have $a \geqslant n(1 + i_n)$. Thus in all cases if $n \neq 0$ and $Av = av$, then

$$\|\Phi_\lambda(u(n))\, v\| \leqslant |n|^{1/2} a^{1/2} \|v\|.$$

Also $\Phi_\lambda(u(0))\, v = \lambda v$. Now argue as in the proof of Proposition 2.1.

To prove (ii) we note that $\Phi_\lambda(d_n) = \Phi_0(d_n) - \lambda\Phi_0(u(n))$ for $n \neq 0$. Since $(\Phi_0, V) \in \mathcal{U}$ with highest weight $(0, 1)$ we see that (i) combined with Proposition 2.1 implies (ii). ∎

We now define a larger class of representations of $\mathfrak{d}$ on V (cf. [C-T]). Given $\lambda, \mu \in \mathbb{C}$, set

$$\Phi_{\lambda,\mu}(d_n) = \Phi_\lambda(d_n) + in\mu\Phi_\lambda(u(n)) - \frac{\mu^2}{2}\delta_{n,0}I.$$

A direct calculation yields the following:

LEMMA 3.3. *The operators $\Phi_{\lambda,\mu}(d_n)$ satsify*

(i) $\Phi_{\lambda,\mu}(d_0) \cdot 1 = -(\lambda^2 + \mu^2)/2.$

(ii) $[\Phi_{\lambda,\mu}(d_n), \Phi_{\lambda,\mu}(d_m)]$
$= (m-n)\,\Phi_{\lambda,\mu}(d_{m+n}) + \delta_{n,-m}(n(n^2-1)/12)(1 + 12\mu^2).$

(iii) $[\Phi_{\lambda,\mu}(d_n), \Phi_\lambda(u(m))] = m\Phi_\lambda(u(m+n)) + in^2\mu\delta_{n,-m}.$

(iv) *If $v, w \in V$ and $d \in \mathfrak{d}$, then*

$$\langle \Phi_{\lambda,\mu}(d)\, v, w \rangle = \langle v, \Phi_{\bar\lambda,\bar\mu}(d^*)\, w \rangle.$$

Thus $(\Phi_{\lambda,\mu}, V)$ is a highest weight representation of $\hat{\mathfrak{d}}$ with highest weight

$$h = -(\lambda^2 + \mu^2)/2, \qquad c = 1 + 12\mu^2.$$

Furthermore, if $\lambda, \mu \in \mathbb{R}$, then $(\Phi_{\lambda,\mu}, V) \in \mathcal{U}$.

As an immediate application of Lemmas 3.2 and 3.3, we have the following result:

PROPOSITION 3.4. *The contravariant form on $L(h, c)$ is positive definite if $c \geqslant 1$ and $h \leqslant (1-c)/24$. Given any compact set Ω of highest weights in this range, there is a fixed neighborhood $\mathcal{M}$ of d_0 in $\mathfrak{d}_\infty$ satisfying the conditions of Theorem 2.3 for all the representations $\pi_{h,c}$ with $(h, c) \in \Omega$. If $X \in \mathcal{M}$, $X = X^*$, and $(h, c) \in \Omega$, then the highest eigenvalue $\mu_0(X)$ of the operator $D(X) = \pi_{h,c}(X)|_{H_1}$ depends continuously on h and c.*

Proof. Let the highest weight (h, c) be given in terms of λ, μ as in Lemma 3.3, with $\lambda, \mu \in \mathbb{R}$. Since μ is arbitrary, c can have any value $\geqslant 1$. Eliminating the parameter μ, we have $h = -\lambda^2/2 + (1-c)/24$. Since λ is arbitrary, the only constraint on h is $h \leqslant (1-c)/24$. Now apply Section **2**(a) to see that $L(h, c) \in \mathcal{U}$ for (h, c) in this range.

The existence of a fixed neighborhood $\mathcal{M}$ of d_0, independent of $(h, c) \in \Omega$, on which the statements in Theorem 2.3 are valid is a consequence of the estimates (i) and (ii) in Lemma 3.2. The key point is that the top-order term $\|d\|_{|t|}\|v\|_{t+1}$ in the estimate (ii) of that lemma is independent of h and c, and the lower order terms are estimated in terms of continuous functions of h and c. The scale of Hilbert spaces H_t is constant, so the perturbation arguments in [G-W; Theorem 3.5] give the continuous dependence of $\mu_0(X)$ on h and c. ▌

Remark. It follows from Kac's results on the Shapavalov form that $L(h, c) \in \mathscr{U}$ for all $h \leqslant 0$ and $c \geqslant 1$ (see [Ka2; p. 123(b)]).

We turn now to the following analytical description of the algebra $\hat{\mathfrak{g}}$: Let $\tilde{\mathfrak{g}}$ be the space of all functions $f\colon S^1 \to \mathbb{C}$ with finite Fourier series. We look upon the element $u(n)$ of $\tilde{\mathfrak{g}}$ as the function $e^{in\theta}$ on S^1. We also consider the skew-symmetric form

$$\Omega(f, g) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{df}{d\theta}(e^{i\theta})\, g(e^{i\theta})\, d\theta.$$

Then $\hat{\mathfrak{g}}$ is just the central extension $\tilde{\mathfrak{g}} \oplus \mathbb{C}z$ with commutation relations defined by Ω:

$$[f, g] = \Omega(f, g)\, z$$

for $f, g \in \tilde{\mathfrak{g}}$. In this realization, the anti-involution on $\hat{\mathfrak{g}}$ is $f^* = \bar{f}$. Thus the elements f of $\tilde{\mathfrak{g}}$ such that $f^* = -f$ are precisely the functions that take purely imaginary values. Clearly

$$\hat{\mathfrak{g}}_\infty = C^\infty(S^1; \mathbb{C}) \oplus \mathbb{C}z$$

with the C^∞ topology.

Recall that $\mathscr{D}$ denotes the group of orientation-preserving diffeomorphisms of S^1. We view elements of $\mathscr{D}$ as smooth, strictly monotone increasing functions ϕ on $\mathbb{R}$ such that $\phi(t + 2\pi) = \phi(t) + 2\pi$ (see [Ha; Sect. 2.3]). Given $\phi \in \mathscr{D}$ and $f \in \tilde{\mathfrak{g}}_\infty$, we define $f^\phi(e^{i\theta}) = f(e^{i\phi(\theta)})$. Then one checks easily that $\Omega(f^\phi, g^\phi) = \Omega(f, g)$. Thus $\mathscr{D}$ acts as automorphisms of $\hat{\mathfrak{g}}_\infty$, fixing the element z. We continue to denote this action on $\hat{\mathfrak{g}}_\infty$ by $x \to x^\phi$.

There is also a natural action of $\mathscr{D}$ on $\mathfrak{d}_\infty$ by

$$X^\phi f = (X(f^{\phi^{-1}}))^\phi$$

for $X \in \mathfrak{d}_\infty$ and $f \in \tilde{\mathfrak{g}}_\infty$. We note that

$$[X^\phi, f^\phi] = X^\phi \cdot f^\phi = [X, f]^\phi,$$

where the commutator is taken in the algebra $\mathfrak{d}_\infty \ltimes \tilde{\mathfrak{g}}_\infty$. However, one must also observe that $\omega(X^\phi, Y^\phi) \neq \omega(X, Y)$ in general, for $\phi \in \mathscr{D}$ and $X, Y \in \mathfrak{d}_\infty$, so that ϕ does not induce an automorphism of the algebra $\hat{\mathfrak{d}}_\infty$.

Using these observations we can define, for any $\phi \in \mathscr{D}$ and $\lambda \in \mathbb{C}$, a new representation Φ_λ^ϕ of $\hat{\mathfrak{g}}_\infty$ on H_∞ by

$$\Phi_\lambda^\phi(x)\, v = \Phi_\lambda(x^\phi)\, v.$$

We can then apply Theorem 2.3 and Lemma 3.2, using the results and techniques of [G-W; Sect. 7] without any essential changes to prove the following (see [G-W; Theorem 7.6]):

THEOREM 3.5. *For all $\lambda \in \mathbb{R}$ there exists a unitary cocycle representation σ_λ of $\mathscr{D}$ on H_0 such that*

(i) *For every $n \geqslant 0$ the map $\mathscr{D} \times H_n \to H_n$ given by $\phi, v \mapsto \sigma_\lambda(\phi)\, v$ is continuous.*

(ii) *If $x \in \hat{\mathfrak{g}}_\infty$ and $\phi \in \mathscr{D}$, then $\sigma_\lambda(\phi)\, \Phi_\lambda(x^\phi) = \Phi_\lambda(x)\, \sigma_\lambda(\phi)$.*

An immediate corollary of Theorem 3.5 is that $\mathscr{D}$ preserves the cohomology class of ω:

COROLLARY 3.6. *If $\phi \in \mathscr{D}$ then there exists $\alpha_\phi \in \mathfrak{d}_\infty{}'$ (continuous dual) such that $\alpha_\phi(d)$ is a jointly continuous function of $\phi \in \mathscr{D}$ and $d \in \mathfrak{d}_\infty$, and*

$$\omega(X^\phi, Y^\phi) = \omega(X, Y) - \alpha_\phi([X, Y]) \tag{3.8}$$

for $X, Y \in \mathfrak{d}_\infty$ and $\phi \in \mathscr{D}$.

Proof. Let $\phi \in \mathscr{D}$ and $X \in \mathfrak{d}_\infty$. Then the operator $\sigma_\lambda(\phi)^{-1} \Phi_\lambda(X)\, \sigma_\lambda(\phi) - \Phi_\lambda(X^\phi)$ commutes with $\Phi_\lambda(y)$ for all $y \in \hat{\mathfrak{g}}_\infty$. By a highest weight argument, using [G-W; Lemma 7.7], we see that this operator must be a scalar multiple $\alpha_\phi(X)$ of the identity. Equation (3.8) is now a simple calculation. The joint continuity of $\alpha_\phi(X)$ follows from the continuity of Φ_λ and σ_λ. ∎

LEMMA 3.7. *Let α_ϕ for $\phi \in \mathscr{D}$ be as in Corollary 3.6. Then α_ϕ is uniquely determined by Eq. (3.8). Furthermore,*

$$\alpha_\phi(d_0) = \frac{1}{48\pi} \int_0^{2\pi} \frac{\phi''(\theta)^2 + \phi'(\theta)^4 - \phi'(\theta)^2}{\phi'(\theta)^3}\, d\theta. \tag{3.9}$$

Proof. The first assertion follows directly from the observation that $[\mathfrak{d}, \mathfrak{d}] = \mathfrak{d}$ and the continuity of α_ϕ. Formula (3.9) is then an elementary but lengthy calculation from (3.8) using the integral formula for ω and the relation

$$2\alpha_\phi(d_0)=\omega(d_1^\phi,\, d_{-1}^\phi).$$

(Recall that $\omega(d_1, d_{-1})=0$.) We omit the details, since it is only the continuous dependence of $\alpha_\phi(d_0)$ on ϕ that will be used in this paper. ∎

Notation. We shall write $\gamma(\phi)$ for $\alpha_\phi(d_0)$.

4. The Integrability Theorem

Let (π, V) be a $\hat{\mathfrak{d}}$ module in the category $\mathscr{U}$, with the highest weight (h, c). Given $\phi\in\mathscr{D}$ and $X\in\mathfrak{d}_\infty$, define

$$\pi^\phi(X)=\pi(X^\phi)+c\alpha_\phi(X), \tag{4.1}$$

as an operator on $H_\infty(\pi)$. From the defining relation (3.8) for α_ϕ, it is a simple calculation to verify the following:

Lemma 4.1. *For $\phi\in\mathscr{D}$, the map $X\mapsto\pi^\phi(X)$ is a projective representation of $\mathfrak{d}_\infty$ on $H_\infty(\pi)$ with cocycle $c\omega$. The contravariant form for (π, V) is also contravariant for π^ϕ.*

Let H_t be the scale of Hilbert spaces associated with (π, V) as in Section 2.

Definition. The module (π, V) is *integrable* if there exists a unitary cocycle representation σ of $\mathscr{D}$ on H_0 such that

(i) For every $n\geqslant 0$ the map $\mathscr{D}\times H_n\to H_n$ given by $\phi, v\mapsto\sigma(\phi)\,v$ is continuous;

(ii) For all $\phi\in\mathscr{D}$ and $X\in\mathfrak{d}_\infty$, one has

$$\sigma(\phi)\,\pi^\phi(X)=\pi(X)\,\sigma(\phi). \tag{4.2}$$

Thus integrability means that the twisted representations π^ϕ of $\mathfrak{d}_\infty$ are all unitarily equivalent, and that the equivalence can be smoothly implemented by a global cocycle representation of $\mathscr{D}$. The main result of this paper is then the following:

Theorem 4.2. *Suppose $h\leqslant 0$ and $c>0$ is such that the irreducible module $L(h, c)$ is unitarizable. Then it is integrable.*

The proof of this theorem will occupy the rest of this section. We start by considering the action of $\mathscr{D}$ on the operator $\pi(d_0)$, where $(\pi, V) \in \mathscr{U}$ with highest weight (h, c). Let $\mathscr{M}$ be the neighborhood of d_0 in $\mathfrak{d}_\infty$ given in Theorem 2.3. Set

$$W_{h,c} = \{\phi \in \mathscr{D} \mid d_0^\phi \in \mathscr{M}\}. \tag{4.3}$$

Then $W_{h,c}$ is a neighborhood of the identity in $\mathscr{D}$. By Theorem 2.3(iii), if $\phi \in W_{h,c}$, then the operator $\pi(d_0^\phi)$ on $H_\infty(\pi)$ has highest eigenvalue $\mu_0(d_0^\phi)$ with multiplicity one. We set

$$\delta_{h,c}(\phi) = \mu_0(d_0^\phi). \tag{4.4}$$

Note that by Section 2(a) this eigenvalue only depends on ϕ and the highest weight (h, c) of (π, V). It can be calculated from any embedding of $L(h, c)$ as the cyclic submodule generated by the highest weight space in some $(\pi, V) \in \mathscr{U}$.

Recall that $\gamma(\phi) = \alpha_\phi(d_0)$. The key step in extending the methods of [G-W; Sect. 7] to arbitrary unitarizable $L(h, c)$ will be to relate $\gamma(\phi)$ to $\delta_{h,c}(\phi)$. We first establish the following properties of $\delta_{h,c}(\phi)$:

LEMMA 4.3. *Assume that $L(h, c) \in \mathscr{U}$. Then the following hold:*

(i) *The function $\phi \to \delta_{h,c}(\phi)$ is continuous on $W_{h,c}$.*

(ii) *If $\Omega \subset \{(h, c) \mid c \geqslant 1, h \leqslant (1-c)/24\}$ is compact, then there is a neighborhood W_Ω of the identity in $\mathscr{D}$ such that $W_\Omega \subset W_{h,c}$ for all $(h, c) \in \Omega$, and $\delta_{h,c}(\phi)$ is a jointly continuous function of (h, c) and ϕ on $\Omega \times W_\Omega$.*

(iii) *If $\phi \in W_{h,c}$ and $\phi^{-1} \in W_{h',c}$, where $h' = \delta_{h,c}(\phi) + c\gamma(\phi)$, then*

$$\delta_{h',c}(\phi) = h - c\gamma(\phi^{-1}). \tag{4.5}$$

(iv) *If $L(h, c)$ is integrable, then*

$$\delta_{h,c}(\phi) = h - c\gamma(\phi) \tag{4.6}$$

for all $\phi \in W_{h,c}$.

Proof. Statement (i) follows from Theorem 2.3 (iii), and (ii) follows from Proposition 3.4.

To prove (iii), let $v_{h,c}(\phi)$ be an eigenvector for $\pi(d_0^\phi)$ with eigenvalue $\delta_{h,c}(\phi)$. Form the cyclic π^ϕ-submodule

$$V_\phi = \pi^\phi(U(\mathfrak{d})) \cdot v_{h,c}(\phi) \tag{4.7}$$

in $H_\infty(\pi)$. Let ρ^ϕ be the restriction of π^ϕ to V_ϕ. Then by Theorem 2.3, (ρ^ϕ, V_ϕ) is in the category $\mathscr{U}$, and from the definition of the twisted representation π^ϕ we calculate that (ρ^ϕ, V_ϕ) has highest weight h'. Hence by Section 2(a), (ρ^ϕ, V_ϕ) is isomorphic to $L(h', c)$.

Now repeat this construction for ϕ^{-1}, starting with the module (ρ^ϕ, V_ϕ); note that by Theorem 2.3(ii) the space $H_\infty(\rho^\phi) \subset H_\infty(\pi)$. By definition, the operator $\rho^\phi(d_0^{\phi^{-1}})$ has highest eigenvalue $\delta_{h',c}(\phi^{-1})$ on $H_\infty(\rho^\phi)$. But

$$\rho^\phi(d_0^{\phi^{-1}}) = \pi(d_0)\,|_{H_\infty(\rho^\phi)} - c\gamma(\phi^{-1}),$$

so by Theorem 2.3(iii) the highest eigenvalue of $\rho^\phi(d_0^{\phi^{-1}})$ is $h - c\gamma(\phi^{-1})$. This proves (iii).

When $L(h, c)$ is integrable, the self-adjoint operators $\pi_{h,c}(d_0^\phi) + \gamma(\phi)\, I$ and $\pi_{h,c}(\phi)$ are unitarily equivalent. This implies (iv). ∎

Definition. A $\mathfrak{S}$ module (π, V) in the category $\mathscr{U}$ with highest weight (h, c) satisfies the *phase-shift condition* (ψ) if Eq. (4.6) holds for all ϕ in some neighborhood of the identity $W \subset W_{h,c}$ in $\mathscr{D}$.

Lemma 4.4. *If $L(h, c) \in \mathscr{U}$ and satisfies (ψ), then $L(h, c)$ is integrable.*

Proof. For $\phi \in W_{h,c}$, let (ρ^ϕ, V_ϕ) be defined as in the proof of Lemma 4.3(iii). Since condition (ψ) holds, this irreducible $\mathfrak{S}$ module has highest weight (h, c), hence is equivalent to $L(h, c)$. Arguing just as in [G-W; Sect. 7], using the uniqueness of the contravariant form on $L(h, c)$, we obtain a unitary cocycle representation σ of $\mathscr{D}$ satisfying the integrability conditions (i) and (ii) (see [G-W; Theorem 7.6]). ∎

Proof of Theorem 4.2. By Lemma 4.4, we only need to show that condition (ψ) is satisfied for all $L(h, c) \in \mathscr{U}$. For this, we shall use properties Section 2(a) and 2(b) of the category $\mathscr{U}$, and the explicit Fock models of Section 3, verifying (ψ) for successively larger sets of highest weights as follows:

(a) The modules (Φ_λ, V) for $\lambda \in \mathbb{R}$ satisfy (ψ). Hence $L(h, 1)$ satisfies (ψ) for all $h \leqslant 0$.

To establish (a), let σ_λ be as in Theorem 3.5. Then by the argument in Corollary 3.6, $\Phi_\lambda(d_0^\phi) + \gamma(\phi) = \sigma_\lambda(\phi)^{-1}\, \Phi_\lambda(d_0)\, \sigma_\lambda(\phi)$ for all $\phi \in \mathscr{D}$. Hence the highest eigenvalue of $\Phi_\lambda(d_0^\phi)$ is $h - \gamma(\phi)$, where $h = -\lambda^2/2$, as claimed.

(b) If $(\pi, V) \in \mathscr{U}$ and n is a positive integer, then (π, V) satisfies (ψ) if and only if $(\pi^{\otimes n}, V^{\otimes n})$ satisfies (ψ).

This is obvious, since the highest eigenvalue of $\pi^{\otimes n}(d_0^\phi)$ is $n\delta_{h,c}(\phi)$ and the highest weight of $(\pi^{\otimes n}, V^{\otimes n})$ is (nh, nc).

(c) If $r > 0$ is rational, and both $L(h, c)$ and $L(rh, rc)$ are in $\mathscr{U}$, then condition (ψ) for $L(h, c)$ is equivalent to condition (ψ) for $L(rh, rc)$.

Write $r = p/q$. Then by (b), (ψ) for $L(h, c)$ implies (ψ) for $L(h, c)^{\otimes p}$, which contains $L(ph, pc)$ as the highest weight cyclic submodule. Hence (ψ) holds for $L(ph, pc)$. Since $L(rh, rc)^{\otimes q}$ has highest weight (ph, pc), it also contains $L(ph, pc)$ as the highest weight cyclic submodule, hence satisfies (ψ). We finally conclude that $L(rh, rc)$ satisfies (ψ) by (b). Reverse the argument by replacing r by $1/r$.

(d) Let $\Sigma = \{(h, c) \in \mathbb{R}^2 \mid c \geqslant 1 \text{ and } h \leqslant (1-c)/24\}$. If $(h, c) \in \Sigma$, then $L(h, c)$ satisfies condition (ψ).

We know by Proposition 3.4 that $L(h, c) \in \mathscr{U}$. First consider the case c rational. By (a) we have $L(h/c, 1)$ satisfying (ψ). Now use (c) to conclude that $L(h, c)$ satisfies (ψ). Since (ψ) is valid for $L(h, c)$ with (h, c) in a dense subset of Σ, we see from Proposition 3.4 that (ψ) holds for all $(h, c) \in \Sigma$, by the continuity of $\delta_{h,c}(\phi)$ and $\gamma(\phi)$ (Note that the neighborhood $\mathscr{M}$ in Proposition 3.4 can be chosen in a locally constant manner.)

(*e*) If $L(h, c) \in \mathscr{U}$ and $h < 0$, then $L(h, c)$ satisfies (ψ).

Indeed, there is a rational number $r > 0$ so that $(rh, rc) \in \Sigma$, where Σ is the sector in (d); see Fig. 1. Now apply (c).

It now only remains to consider the modules $L(0, c) \in \mathscr{U}$, with $c > 0$. For this case we argue by contradiction, as follows:

Suppose (ψ) fails, for some $\phi \in W_{0,c}$. Let $(\rho^{\phi}, V_{\phi}) \in \mathscr{U}$ be as in (4.7). This module is isomorphic to $L(h', c)$, where $h' = \delta_{0,c}(\phi) - c\gamma(\phi)$. We are assuming that (ψ) fails for ϕ, so that $h' \neq 0$. Since $L(h', c) \in \mathscr{U}$, we see from Lemma 1.1 that $h' < 0$. Thus $L(h', c)$ is integrable, by (e) and Lemma 4.4. In particular, the operator $\pi_{h',c}(d_0^{\alpha})$ is unitarily equivalent to $\pi_{h',c}(d_0) - c\gamma(\alpha)$ for all $\alpha \in \mathscr{D}$, and hence $\delta_{h',c}(\alpha) = h' - c\gamma(\alpha)$. Taking $\alpha = \phi^{-1}$ and comparing with (4.5), we find that $h' = 0$, which is a contradiction. This completes the proof of Theorem 4.2. ∎

Concluding Remarks

Let $(\pi_{h,c}, L(h, c)) \in \mathscr{U}$, and let σ be a unitary cocycle representation of $\mathscr{D}$ which "integrates" $\pi_{h,c}$ in the sense of the definition at the beginning of this section. By a simple modification of the argument of [G-W; Lemma 7.7

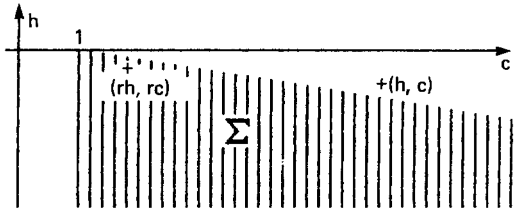

FIG. 1. Region Σ of unitary from Fock models (phase shift at (h, c) outside Σ determined by phase shift at point (rh, rc) in Σ).

and Theorem 7.8], one concludes that σ is unique up to multiplication by a continuous function from $\mathscr{D}$ to S^1. (Replace the subalgebra $\hat{\mathfrak{n}}$ of $\hat{\mathfrak{g}}$ in [G-W; Lemma 7.7] by the subalgebra $\mathfrak{n}$ of $\hat{\mathfrak{d}}$ defined in Sect. 1.) Using some results of Segal and Bott, as in [G-W; Theorem 7.10], one may normalize σ so that

$$\sigma(\exp X) = e^{\pi_{h,c}(X)} \tag{4.8}$$

for all real C^∞ vector fields X on S^1. Here $\exp X$ is the diffeomorphism of S^1 generated by X, while the right side of (4.8) is defined via the spectral theorem, by Corollary 2.2.

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